

LIE POINT SYMMETRY DATA AUGMENTATION FOR NEURAL PDE SOLVERS

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Neural PDE solver 'chicken-and-egg problem'

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- Need cheap ways to enlarge our datasets
- **Solution:** Data augmentation! But how?

PDE formulation

PDE specifies relationship between *solution* $\mathbf{u} : X \rightarrow \mathbb{R}^n$ and its derivatives $f(\mathbf{u}_x, \mathbf{u}_{xx}, \dots, \mathbf{u}_{nx})$ at all points $\mathbf{x} \in X$ as

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Example

1 + 1 dimensional Heat equation in space and time:

$$\Delta(\mathbf{x}, u^{(2)}_{\mathbf{x}}) = u_t - \alpha u_{xx} = 0$$

- Pointwise transformation:

$$(\mathbf{x}, \mathbf{u}_{j_{\mathbf{x}}}) \xrightarrow{g} (g\mathbf{x}, g\mathbf{u}_{j_{\mathbf{x}}}), \quad \text{for all } g \text{ and } (\mathbf{x}, \mathbf{u})$$

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Set of g mappings solutions $(\mathbf{x}, \mathbf{u}_{j_x})$ to solutions $(g\mathbf{x}, g\mathbf{u}_{j_x})$

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Cauchy-Kovalevskaya theorem

f, g_1, \dots, g_d exhaustive if PDE analytic in its arguments

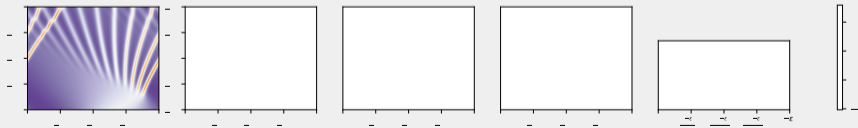
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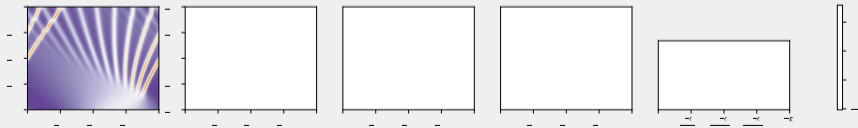


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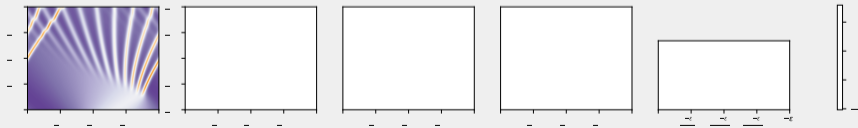


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2. Sample $\theta_i \sim p_i(\theta_i)$
3. Compute $g = g_1(\theta_1) \circ g_2(\theta_2) \circ \dots \circ g_d(\theta_d)$
4. Transform samples: $(x; u_{j,x}) \rightarrow (gx; gu_{j,x})$

Lie point symmetry data augmentation

1. Find all one-parameter symmetries of f : $g_1; g_2; \dots; g_n$
2. Sample $p_i(x)$
3. Compute $g = g_1(x_1) g_2(x_2) \dots g_d(x_d)$
4. Transform samples: $(x; u_{j,x}) \rightarrow (gx; gu_{j,x})$
5. Reinterpolate to grid: $u(x) \rightarrow u^0(x)$

Different PDEs, different symmetries

- Korteweg-de Vries equation:

$$((x; t); u^{(3)}) = u_t + uu_x + u_{xxx} = 0$$

- Kuramoto-Shivashinsky equation:

$$((x; t); u^{(4)}) = u_t + uu_x + u_{xx} + u_{xxxx} = 0$$

- Burgers' equation:

$$((x; t); u^{(2)}) = u_t + uu_x \quad u_{xx} = 0$$

Equation	g_1	g_2	g_3	g_4	g
KdV	$(x; t + \epsilon; u)$	$(x + \epsilon; t; u)$	$(x + \epsilon; t; u + \epsilon)$	$(e^{-\epsilon} x; e^3 \epsilon t; e^{-2} u)$	
KS	$(x; t + \epsilon; u)$	$(x + \epsilon; t; u)$	$(x + \epsilon; t; u + \epsilon)$		
Burgers'	$(x; t + \epsilon; u)$	$(x + \epsilon; t; u)$	$(x; t; u + \epsilon)$	$(e^{-\epsilon} x; e^2 \epsilon t; u)$	$u; t; 2 \log(1 - \epsilon) e^{\frac{1}{2}u} + (\epsilon) e^{\frac{1}{2}}$

Different PDEs

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Different training methods

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Different models

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Combining symmetries

Long rollouts

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CONCLUSION

- Mathematically-grounded treatment of data augmentation for neural PDE solvers
- LPSDA comprises the full set of solution-preserving, pointwise, continuous data transformations for any (analytic) PDEs
- LPSDA improves neural PDE solver sample complexity to over an order of magnitude
- Solver-agnostic & PDE-agnostic

PAPER: LIE POINT SYMMETRY DATA AUGMENTATION FOR
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ARXIV:2202.07643

CODE: <https://github.com/zhengzhang1999/LPSDA>